Some Groups Generated by Transvections

By

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Let V be a vector space of dimension $n \geqq 2$ over a field K. For each pair of subspaces $P \subseteqq H$ of dimension 1 and $n-1$ respectively the subgroup of $SL(V)$ generated by those transvections τ with $H = \operatorname{Ker}(\tau - 1)$, $P = \operatorname{Im}(\tau - 1)$ will be denoted by $X(P, H)$. Drawing from the language of Lie theory, JOHN THOMPSON has christened these *subgroups of root type*, and he has asked which subgroups of $SL(V)$ having only 1 as a normal unipotent subgroup are generated by subgroups of root type. In this note we give the following partial answer.

Theorem. *Suppose* $K \neq \boldsymbol{F}_2$, $\dim V \geqq 2$, *and that* G *is a subgroup of* $SL(V)$ *which is generated by subgroups of root type. Suppose also that* 1 *is the only normal unipotent subgroup of* G. *Then for some* $s \geqq 1$, $V = V_0 \oplus V_1 \oplus \cdots \oplus V_s$, $G = G_1 \times \cdots \times G_s$, *and*

(a) *The* V_i *are stable for the* G_j.

(b) $G_i \mid V_j = 1$ *if* $i \neq j$.

(c) $G_i \mid V_i = SL(V_i)$ *or* $Sp(V_i)$.

For $K = \boldsymbol{F}_2$ there are many other examples, but we have been unable to construct a definitive list.

The proof of the theorem will be made by examining the action of G on the set of subspaces of V. We begin by describing the terminology we shall use. For the time being K is *any* field and we suppose only that $\dim V \geqq 2$ and $G \subseteqq SL(V)$ is generated by subgroups of root type. (We take this to mean $G \neq 1$ — G contains subgroups of root type.) This hypothesis will *always* be in force. Others will be explicitly stated when used. We say P is a center (for G) if, for some H, $X(P, H) \subseteqq G$; likewise H is an *axis* (for G). We will also say H is an axis for P and P is a center for H. For each center P, $a(P)$ is the intersection of the axes for P; dually $c(H)$ is the sum of the centers for H. $\mathfrak{C}$ is the set of centers for G and $\mathfrak{A}$ is the set of axes for G. If U is a subspace of V, G_U is the stabilizer of U in the action of G on the subspaces of V. Finally, since we are primarily concerned with subspaces rather than vectors and point is easier to say than one-dimensional subspace, we shall use the term *point* for one-dimensional subspace.

Lemma 1. *If* $\dim V = 2$ *and* $|\mathfrak{C}| > 1$ *then* $G = SL(V)$.

*) Work supported in part by a grant from the National Science Foundation.

Proof. Clear.

Lemma 2. *Let P and Q be centers with* $\dim a(P) \leqq \dim a(Q)$ *and* $P \nsubseteq a(Q)$. *Then* $Q \nsubseteq a(P)$ *and if* $U \supseteqq P + Q$ *then* $G_{U,P+Q} | P + Q$ *contains* $SL(P + Q)$. *In particular every point on* $P + Q$ *is a center.*

Proof. If $Q \subseteqq a(P)$, then Q and so $a(Q)$ is fixed by all $X(P, H) \subseteqq G$. Since $P \nsubseteq a(Q)$, $a(Q)$ is on all axes of $P - a(Q) \subseteqq a(P)$. Then $a(Q) = a(P)$ against $P \nsubseteq a(Q)$. Thus G contains subgroups $X(P, H)$, $X(Q, K)$ with $P \nsubseteq K$, $Q \nsubseteq H$. If $U \supseteqq P + Q$, each of these subgroups is in $G_{U,P+Q}$. The second conclusion then follows from Lemma 1.

We can now give the first approximation to the theorem.

Lemma 3. *Suppose* $\mathfrak{C}$ *consists of all points,* $\mathfrak{A}$ *consists of all hyperplanes, and* G *is transitive on these two sets. Then either* $G = SL(V)$ *or* $G = Sp(V)$.

Proof. Suppose first that each P has a unique axis δP and each H has a unique center δH. Then $a(P) = \delta P$, $c(H) = \delta H$. By Lemma 2, $P \subseteqq \delta Q$ if and only if $Q \subseteqq \delta P$. That is, $P \subseteqq H$ if and only if $\delta H \subseteqq \delta P$ and δ extends to a null polarity on the set of subspaces of V. If $X(P, \delta P) \subseteqq G$ and $\sigma \in G$ then $\sigma X(P, \delta P) \sigma^{-1} = X(\sigma P, \sigma \delta P) \subseteqq G$. Hence $\sigma \delta P = \delta \sigma P$ and G is in the group of δ. It is clear that G contains *all* transvections commuting with δ and hence $G = Sp(V)$.

Next suppose $a(P)$ is not a hyperplane. Let H be an axis for P and choose $Q \subseteqq H$, $Q \nsubseteq a(P)$. Then by Lemma 2, we have $\sigma \in G_H$ with $\sigma P = Q$. Hence every point of H which is not on $a(P)$ is a center for H and consequently $c(H) = H$. Then by duality $a(P) = P$ for all points P. Now the above shows that all points of H are centers for H. Thus G contains all transvections and $G = SL(V)$.

Finally if we start with the assumption that $c(H)$ is not a point and dualize the above we again get $G = SL(V)$.

Remark 1. If the field K is finite one can draw the same conclusion from the (formally) weaker hypothesis that G is irreducible and each P in V is the center for some transvection in G.

Remark 2. A corollary of the lemma is that $Sp(V)$ is a maximal subgroup of $SL(V)$ (except, of course, when $\dim V = 2$).

Lemma 4. *Suppose G is transitive on* $\mathfrak{C}$. *If P, S are distinct members of* $\mathfrak{C}$ *with* $S \subseteqq a(P)$ *there is a center Q with* $P \nsubseteq a(Q)$ *and* $S \nsubseteq a(Q)$.

Proof. If not then for all T in the G_P-orbit of S we have $P \nsubseteq a(Q)$ implies $T \subseteqq a(Q)$. Choose $X(R, K) \subseteqq G$. If $P \subseteqq a(R)$ then $X(R, K) \subseteqq G_P$ and $X(R, K)$ fixes the G_P-orbit of S. If $P \nsubseteq a(R)$ then $X(R, K)$ fixes each member of the G_P-orbit of S. Since G is generated by the $X(R, K)$ we contradict the transitivity of G on $\mathfrak{C}$.

Lemma 5. *Take* $K \neq \mathbf{F}_2$ *and suppose G is transitive on* $\mathfrak{C}$. *If* $P, S \in \mathfrak{C}$ *then each point on* $P + S$ *is in* $\mathfrak{C}$.

Proof. Since G is transitive on $\mathfrak{C}$, all $a(P)$ have the same dimension and hence by Lemma 2, $P \nsubseteq a(S)$ if and only if $S \nsubseteq a(P)$ and we have nothing to prove if $S \nsubseteq a(P)$. Suppose $S \neq P$, $S \subseteqq a(P)$, and choose $Q \in \mathfrak{C}$ so $S \nsubseteq a(Q)$ and $S \nsubseteq a(P)$ (Lemma 4).

Set $U = P + Q + S$. G contains the subgroups $X(P, H)$, $X(S, J)$, and $X(Q, K)$ with $P \nsubseteq K$, $Q \nsubseteq H$, and $Q \nsubseteq J$. Since $P + S \subseteqq a(P)$, $P + S = a(P) \cap U = H \cap U$. Since $X(P, H)$ and $X(Q, K)$ fix U, $a(P) \cap U$ and $a(Q) \cap U$ have the same dimension and therefore $a(Q) \cap U = K \cap U$ and $S \nsubseteq K$. We want to determine the group induced on U by $X(P, H)$, $X(S, J)$ and $X(Q, K)$. Although the result is wellknown and can be easily obtained synthetically, we sketch the argument using matrices. Set $R = U \cap H \cap K$ and choose a base for U consisting of x_1, x_2, x_3 from R, P, Q respectively. The group induced on U by $X(P, H)$ and $X(Q, K)$ will then be represented by the group of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & u_1 & u_2 \\ 0 & v_1 & v_2 \end{pmatrix}$$

where

$$\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}$$

is an arbitrary member of $SL_2(K)$. The group induced by $X(S, J)$ will be represented by the matrices

$$\begin{pmatrix} 1 & 0 & ta \\ 0 & 1 & tu \\ 0 & 0 & 1 \end{pmatrix}$$

where $au \neq 0$ and t is arbitrary in K. Putting these two groups together we see that G is transitive on the points of U other than R so that all such points are centers for G. We also see that if L is a complement for R in $H \cap K$ and $J \supseteqq L$ then R is also a center for G and we are done. Thus we suppose $J \cap L$ is a hyperplane of L, choose a complement, $T = \langle x_4 \rangle$, and with respect to the base $\{x_1, x_2, x_3, x_4\}$ for $U + T$ we again look at the matrices determined by our three groups — now acting on $U + T$. As before $X(P, H)$ and $X(Q, K)$ will yield the group of

$$D(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where A is arbitrary in $SL_2(K)$, and from $X(S, J)$ we get a matrix

$$M = \begin{pmatrix} 1 & \alpha & c \\ 0 & Z & \beta \\ 0 & 0 & 1 \end{pmatrix}$$

where

$$\alpha = (0\ a), \quad Z = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} b \\ 0 \end{pmatrix}$$

and $abc \neq 0$. Set

$$N(A) = D(A)\, D(Z^{-1})\, M\, D(A^{-1}) = \begin{pmatrix} 1 & \alpha A^{-1} & c \\ 0 & 1 & A\beta \\ 0 & 0 & 1 \end{pmatrix}.$$

Then for $A, B \in SL_2(K)$ we have

$$D(Z^{-1})\, M\, N(A)\,(N(B))^{-1} = \begin{pmatrix} 1 & \alpha(1+A^{-1}) & 2c+\alpha A\beta \\ 0 & 1 & (A+1)\beta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\alpha B^{-1} & -c \\ 0 & 1 & -B\beta \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & \alpha(1+A^{-1}-B^{-1}) & c+\alpha A\beta-\alpha(1+A^{-1})\,B\beta \\ 0 & 1 & (A+1-B)\,\beta \\ 0 & 0 & 1 \end{pmatrix}.$$

Take

$$A = \begin{pmatrix} -1 & 1/x \\ -x & 0 \end{pmatrix} \quad \text{and} \quad B = 1 + A = \begin{pmatrix} 0 & 1/x \\ -x & 1 \end{pmatrix}$$

so

$$A^{-1} = -B, \quad B^{-1} = -A, \quad 1 + A^{-1} - B^{-1} = 1 + A^{-1} + A = 0, \quad A - B = -1.$$

Since $\alpha\beta = 0$ our product is

$$\begin{pmatrix} 1 & 0 & c-\alpha A^{-1}\beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since

$$(0\ a)\begin{pmatrix} 0 & -1/x \\ x & -1 \end{pmatrix}\begin{pmatrix} b \\ 0 \end{pmatrix} = abx$$

it suffices to choose $x \neq c/ab$ and this can be done if $K \neq \mathbf{F}_2$. Thus R is a center and the proof of the lemma is complete.

Lemma 6. *Suppose G has no normal unipotent subgroups save* 1 *and that G is transitive on $\mathfrak{A}$, then G is transitive on $\mathfrak{C}$.*

Proof. Choose a center, P, with $a(P)$ maximal. By Lemma 2, G has one orbit of centers containing P and all centers off $a(P)$. If there is another orbit, its members all lie on $a(P)$ and consequently lie on all axes. If Q is a member of this orbit, Q is fixed by G and

$$\langle X(Q, K) \mid K \text{ an axis for } Q \rangle$$

is a normal unipotent subgroup not 1.

Lemma 7. *Suppose $K \neq \mathbf{F}_2$ and G contains no normal unipotent subgroups save* 1. *Suppose further that G is transitive on $\mathfrak{C}$. Then $V = C \oplus A$ where C and A are stable for G; $G \mid A = 1$, and $G \mid C = SL(C)$ or $Sp(C)$.*

Proof. Let C be the sum of the members of $\mathfrak{C}$ and let A be the intersection of the members of $\mathfrak{A}$. These spaces admit G. Suppose $P \subseteqq A \cap C$. By Lemma 5, $P \in \mathfrak{C}$. But $P \subseteqq A$ implies that P is fixed by G and this contradicts the assumption that G has only 1 as a normal unipotent subgroup. Dually $C + A = V$. Certainly $G \mid A = 1$. Let H_1 be a hyperplane of C, so $H_1 + A = H$ is a hyperplane of V and $H \cap C = H_1$. Again by the dual of Lemma 5, H is an axis for G. Suppose $X(P, H) \subseteqq G$; then

$X(P, H)|C$ is a subgroup of root type in $SL(C)$ having P as center, H_1 as axis. Thus every hyperplane of C is an axis for $G|C$ and by the dual of Lemma 6 $G|C$ is transitive on the hyperplanes of C since G is transitive on C. Similarly each point of C is a center for $G|C$ and $G|C$ is transitive on these centers. The conclusion now follows from Lemma 3.

We now can give a proof of the theorem. For this it suffices to suppose V indecomposable. Choose an orbit of centers, $\mathfrak{O}_1$, such that $P \in \mathfrak{O}_1$ implies $a(P)$ minimal. Set

$$G_1 = \langle X(P, H) \mid P \in \mathfrak{O}_1 \rangle,$$
$$G_1^* = \langle X(Q, K) \mid Q \notin \mathfrak{O}_1 \rangle.$$

These are normal subgroups of G and by Lemma 2 if $P \in \mathfrak{O}_1$ then G_1^* fixes P and $G_1^*|P = 1$. Thus $G_1 \cap G_1^*$ is a normal unipotent subgroup so $G_1 \cap G_1^* = 1$. Since $G = G_1 \times G_1^*$, G_1 has no normal unipotent subgroups. Also G_1 is transitive on its set of centers — $\mathfrak{O}_1$. If C_1 is the sum of the centers of G_1 and A_1 is the intersection of the axes of G_1, then from Lemma 7 $V = C_1 \oplus A_1$. But C_1 and A_1 admit G_1^* so $A_1 = 0$, $G_1^* = 1$ and $G = SL(V)$ or $Sp(V)$.

Added in proof: Since this paper was submitted for publication, the following paper, containing different but related results obtained by the same basic strategy, has appeared:

F. C. Piper, On elations of finite projective spaces of odd order. J. London Math. Soc. 41, 641–648 (1966).

Eingegangen am 27. 6. 1966

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